A REALIZATION THEOREM FOR MODULES OF CONSTANT JORDAN TYPE AND VECTOR BUNDLES

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Abstract. Let E be an elementary abelian p-group of rank r and let k be a field of characteristic p. We introduce functors $\mathcal{F}_i$ from finitely generated kE-modules of constant Jordan type to vector bundles over projective space $\mathbb{P}^{r-1}$. The fibers of the functors $\mathcal{F}_i$ encode complete information about the Jordan type of the module.

We prove that given any vector bundle $\mathcal{F}$ of rank s on $\mathbb{P}^{r-1}$, there is a kE-module M of stable constant Jordan type $[1]^s$ such that $\mathcal{F}_1(M) \cong \mathcal{F}$ if $p = 2$, and such that $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$ if p is odd. Here, $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$ is the Frobenius map. We prove that the theorem cannot be improved if p is odd, because if M is any module of stable constant Jordan type $[1]^s$ then the Chern numbers $c_1, \ldots, c_{p-2}$ of $\mathcal{F}_1(M)$ are divisible by p.

1. Introduction

The class of modules of constant Jordan type was introduced by Carlson, Friedlander and the second author [5], and then consequently studied in [1, 2, 4, 6, 7, 8]. The connection between modules of constant Jordan type and algebraic vector bundles on projective varieties was first observed and developed by Friedlander and the second author in [7] in the general setting of an arbitrary infinitesimal group scheme. In the present paper, we study this connection for an elementary abelian p-group.

Let k be a field of characteristic p and let E be an elementary abelian p-group of rank r. We define functors $\mathcal{F}_i$ $(1 \leq i \leq p)$ from finitely generated kE-modules of constant Jordan type to vector bundles on projective space $\mathbb{P}^{r-1}$, capturing the sum of the socles of the length i Jordan blocks. The following is the main theorem of this paper.

Theorem 1.1. *Given any vector bundle $\mathcal{F}$ of rank s on $\mathbb{P}^{r-1}$, there exists a finitely generated kE-module M of stable constant Jordan type $[1]^s$ such that*

(i) *if $p = 2$, then $\mathcal{F}_1(M) \cong \mathcal{F}$.*
(ii) *if p is odd, then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, the pullback of $\mathcal{F}$ along the Frobenius morphism $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$.*

The kE-modules produced this way are usually large. For example, in [1], the first author showed how to produce a finitely generated kE-module M of constant Jordan type such that $\mathcal{F}_2(M)$ is isomorphic to the rank two Horrocks–Mumford bundle on $\mathbb{P}^4$. In this case, the construction used to prove our main theorem produces a module M of dimension many hundred times p^5 plus two such that $\mathcal{F}_1(M) \cong F^*(\mathcal{F}_{\mathsf{HM}})$, whereas the construction in [1] produces a module of dimension $30p^5$ of stable constant Jordan type $[p-1]^{30}[2]^2[1]^{26}$ such that applying $\mathcal{F}_2$ gives $\mathcal{F}_{\mathsf{HM}}(-2)$.

The theorem for $p = 2$ may be thought of as a version of the Bernstein–Gelfand–Gelfand correspondence [3], since the group algebra of an elementary abelian 2-group in characteristic

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two is isomorphic to an exterior algebra. But for p odd it says something new and interesting. In particular, it is striking that the p odd case of Theorem 1.1 cannot be strengthened to say that $\mathcal{F}_1(M) \cong \mathcal{F}$. The following theorem, which is proved in Section 5, gives limitations on the vector bundles appearing as $\mathcal{F}_1(M)$ with M of stable constant Jordan type $[1]^s$.

Theorem 1.2. *Suppose that M has stable constant Jordan type $[1]^s$. Then p divides the Chern numbers $c_m(\mathcal{F}_1(M))$ for $1 \leq m \leq p-2$.*

The paper is organized as follows. In Section 2 we give basic definitions of the functors $\mathcal{F}_i$ and show that applied to modules of constant Jordan type, they produce algebraic vector bundles. Section 3 analyzes behavior of the functors $\mathcal{F}_i$ with respect to Heller shifts and duals. This analysis plays a key role in the proof of our main theorem. Theorems 1.1 and 1.2 are proved in Sections 4 and 5 respectively.

2. Definition of the functors $\mathcal{F}_i$

Let k be a perfect field of characteristic p. Let $E = \langle g_1, \dots, g_r \rangle$ be an elementary abelian p-group of rank r, and set $X_i = g_i - 1 \in kE$ for $1 \leq i \leq r$. Let $J(kE) = \langle X_1, \dots, X_r \rangle$ be the augmentation ideal of kE. The images of $X_1, \dots, X_r$ form a basis for $J(kE)/J^2(kE)$, which we think of as affine space $\mathbb{A}^r_k$ over k. Let K/k be a field extension. If $0 \neq \alpha = (\lambda_1, \dots, \lambda_r) \in \mathbb{A}^r_K$, we define

$$X_\alpha = \lambda_1 X_1 + \cdots + \lambda_r X_r \in KE.$$

This is an element of $J(KE)$ satisfying $X_\alpha^p = 0$. If M is a finitely generated kE-module then X_α acts nilpotently on $M_K = M \otimes K$, and we can decompose M_K into Jordan blocks. They all have eigenvalue zero, and length between 1 and p. We say that M has *constant Jordan type* $[p]^{a_p} \dots [1]^{a_1}$ if there are a_p Jordan blocks of length p, ..., a_1 blocks of length 1, independently of choice of α. Since a_p is determined by $a_{p-1}, \dots, a_1$ and the dimension of M, we also say that M has *stable constant Jordan type* $[p-1]^{a_{p-1}} \dots [1]^{a_1}$. Note that the property of having *constant Jordan type* and the type itself do not depend on the choice of generators $\langle g_1, \dots, g_r \rangle$ (see [5]).

We write $k[Y_1, \dots, Y_r]$ for the coordinate ring $k[\mathbb{A}^r]$, where the Y_i are the linear functions defined by $Y_i(X_j) = \delta_{ij}$ (Kronecker delta). We write $\mathbb{P}^{r-1}$ for the corresponding projective space. Let $\mathcal{O}$ be the structure sheaf on $\mathbb{P}^{r-1}$. If $\mathcal{F}$ is a sheaf of $\mathcal{O}$-modules and $j \in \mathbb{Z}$, we write $\mathcal{F}(j)$ for the jth Serre twist $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{O}(j)$. If M is a finitely generated kE-module, we write $\widetilde{M}$ for the trivial vector bundle $M \otimes_k \mathcal{O}$, so that $\widetilde{M}(j) = M \otimes_k \mathcal{O}(j)$. Friedlander and the second author [7, §4] define a map of vector bundles $\theta_M \colon \widetilde{M} \to \widetilde{M}(1)$ by the formula

$$\theta_M(m \otimes f) = \sum_{i=1}^r X_i(m) \otimes Y_i f.$$

By abuse of notation we also write θ_M for the twist $\theta_M(j) \colon \widetilde{M}(j) \to \widetilde{M}(j+1)$. With this convention we have $\theta_M^p = 0$.

We define functors $\mathcal{F}_{i,j}$ for $0 \leq j < i \leq p$ from finitely generated kE-modules to coherent sheaves on $\mathbb{P}^{r-1}$ by taking the following subquotients of $\widetilde{M}$:

$$\mathcal{F}_{i,j}(M) = \frac{\operatorname{Ker} \theta_M^{j+1} \cap \operatorname{Im} \theta_M^{i-j-1}}{(\operatorname{Ker} \theta_M^{j+1} \cap \operatorname{Im} \theta_M^{i-j}) + (\operatorname{Ker} \theta_M^{j} \cap \operatorname{Im} \theta_M^{i-j-1})}$$

We then define

$$\mathcal{F}_i(M) = \mathcal{F}_{i,0}(M) = \frac{\operatorname{\mathsf{Ker}} \theta_M \cap \operatorname{\mathsf{Im}} \theta_M^{i-1}}{\operatorname{\mathsf{Ker}} \theta_M \cap \operatorname{\mathsf{Im}} \theta_M^{i}}$$

For a point $0 \neq \alpha \in \mathbb{A}^r$ and the corresponding operator $X_\alpha \colon M \to M$, we also define

$$\mathcal{F}_{i,\alpha}(M) = \frac{\operatorname{\mathsf{Ker}} X_\alpha \cap \operatorname{\mathsf{Im}} X_\alpha^{i-1}}{\operatorname{\mathsf{Ker}} X_\alpha \cap \operatorname{\mathsf{Im}} X_\alpha^{i}}$$

Note that $\mathcal{F}_{i,\alpha}(M)$ is evidently well-defined for $\bar{\alpha} \in \mathbb{P}^{r-1}$.

In the next Proposition we show that functors $\mathcal{F}_i$ take modules of constant Jordan type to algebraic vector bundles (equivalently, locally free sheaves), and that they commute with specialization.

Proposition 2.1.

(1) *Let M be a kE-module of constant Jordan type $[p]^{a_p} \dots [1]^{a_1}$. Then the sheaf $\mathcal{F}_i(M)$ is locally free of rank a_i.*

(2) *Let $f \colon M \to N$ be a map of modules of constant Jordan type. For any point $\bar{\alpha} = [\lambda_1 : \dots : \lambda_r] \in \mathbb{P}^{r-1}$ with residue field $k(\bar{\alpha})$ we have a commutative diagram*

$$\begin{tikzcd}
\mathcal{F}_i(M) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[r, "\mathcal{F}_i(f)"] \arrow[d, "\simeq"] & \mathcal{F}_i(N) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[d, "\simeq"] \\
\mathcal{F}_{i,\alpha}(M) \arrow[r] & \mathcal{F}_{i,\alpha}(N)
\end{tikzcd}$$

Proof. (1). Since the module M is fixed throughout the proof, we shall use θ to denote θ_M.

Note that $\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i = \operatorname{\mathsf{Ker}} \{\theta \colon \operatorname{\mathsf{Im}} \theta^i \to \operatorname{\mathsf{Im}} \theta^{i+1}\}$. Hence, we have a short exact sequence

$$0 \longrightarrow \operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i \longrightarrow \operatorname{\mathsf{Im}} \theta^i \xrightarrow{\theta} \operatorname{\mathsf{Im}} \theta^{i+1} \longrightarrow 0 \tag{2.1.1}$$

Since M has constant Jordan type, $\operatorname{\mathsf{Im}} \theta^i$ is locally free by [7, 4.13]. Therefore, specialization of the sequence (2.1.1) at any point $\bar{\alpha} = [\lambda_1 : \dots : \lambda_r]$ of $\mathbb{P}^{r-1}$ yields a short exact sequence of vector spaces

$$0 \to (\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i) \otimes_{\mathcal{O}} k(\bar{\alpha}) \to \operatorname{\mathsf{Im}} \theta^i \otimes_{\mathcal{O}} k(\bar{\alpha}) \to \operatorname{\mathsf{Im}} \theta^{i+1} \otimes_{\mathcal{O}} k(\bar{\alpha}) \to 0. \tag{2.1.2}$$

By [7, 4.13], $\operatorname{\mathsf{Im}} \theta^i \otimes_{\mathcal{O}} k(\bar{\alpha}) \simeq \operatorname{\mathsf{Im}} \{X_\alpha^i \colon M \to M\}$. In particular, the dimension of fibers of $\operatorname{\mathsf{Im}} \theta^i$ is constant and equals $\sum\limits_{j=i+1}^{p} a_j(j-i)$. We can rewrite the sequence (2.1.2) as

$$0 \longrightarrow (\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i) \otimes_{\mathcal{O}} k(\bar{\alpha}) \longrightarrow \operatorname{\mathsf{Im}} X_\alpha^i \xrightarrow{X_\alpha} \operatorname{\mathsf{Im}} X_\alpha^{i+1} \longrightarrow 0\,.$$

Hence the fiber of $\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i$ at a point $\bar{\alpha}$ equals $\operatorname{\mathsf{Ker}} X_\alpha \cap \operatorname{\mathsf{Im}} X_\alpha^i$. In particular, $\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i$ has fibers of constant dimension, equal to

$$\sum_{j=i+1}^{p} a_j(j-i) - \sum_{j=i+2}^{p} a_j(j-i-1) = \sum_{j=i+1}^{p} a_j.$$

Applying [7, 4.10] (see also [10, V. ex. 5.8]), we conclude that $\operatorname{\mathsf{Ker}} \theta \cap \operatorname{\mathsf{Im}} \theta^i$ is locally free of rank $\sum\limits_{j=i+1}^{p} a_j$.

Consider the short exact sequence that defines $\mathcal{F}_i(M)$:

$$0 \longrightarrow \mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^i \longrightarrow \mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i-1} \longrightarrow \mathcal{F}_i(M) \longrightarrow 0.$$

Specializing at $\bar{\alpha}$, we get

$$\begin{tikzcd}
(\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^i) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[r] \arrow[d, equal] & (\mathsf{Ker}\,\theta \cap \mathsf{Im}\,\theta^{i-1}) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[r] \arrow[d, equal] & \mathcal{F}_i(M) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[r] \arrow[d, equal] & 0 \\
\mathsf{Ker}\,X_\alpha \cap \mathsf{Im}\,X_\alpha^i \arrow[r] & \mathsf{Ker}\,X_\alpha \cap \mathsf{Im}\,X_\alpha^{i-1} \arrow[r] & \mathcal{F}_i(M) \otimes_{\mathcal{O}} k(\bar{\alpha}) \arrow[r] & 0
\end{tikzcd}$$

The first arrow of the bottom row is clearly an injection. Hence,

$$\dim(\mathcal{F}_i(M) \otimes_{\mathcal{O}} k(\bar{\alpha})) = \sum_{j=i}^{p} a_j - \sum_{j=i+1}^{p} a_j = a_i$$

for any point $\alpha \in \mathbb{P}^{r-1}$. Applying [7, 4.10] again, we conclude that $\mathcal{F}_i(M)$ is locally free (of rank a_i).

Statement (2) follows immediately by applying the last diagram to both M and N. $\square$

Lemma 2.2. *$\widetilde{M}$ has a filtration in which the filtered quotients are isomorphic to $\mathcal{F}_{i,j}(M)$ for $0 \leq j < i \leq p$.*

Proof. We consider two filtrations on $\widetilde{M}$, the "kernel filtration" and the "image filtration":

$$0 \subset \mathsf{Ker}\,\theta_M \subset \ldots \subset \mathsf{Ker}\,\theta_M^{p-1} \subset \widetilde{M}$$
$$0 = \mathsf{Im}\,\theta_M^p \subset \mathsf{Im}\,\theta_M^{p-1} \subset \ldots \subset \mathsf{Im}\,\theta_M \subset \mathsf{Im}\,\theta_M^0 = \widetilde{M}$$

To simplify notation, we set $\mathcal{K}_j = \mathsf{Ker}\,\theta_M^i$ and $\mathcal{I}_i = \mathsf{Im}\,\theta_M^{p-i}$. Using the standard refinement procedure, we refine the kernel filtration by the image filtration:

$$\mathcal{K}_j \subset (\mathcal{K}_{j+1} \cap \mathcal{I}_1) + \mathcal{K}_j \subset \ldots \subset (\mathcal{K}_{j+1} \cap \mathcal{I}_\ell) + \mathcal{K}_j \subset (\mathcal{K}_{j+1} \cap \mathcal{I}_{\ell+1}) + \mathcal{K}_j \subset \ldots \subset \mathcal{K}_{j+1}$$

For any three sheaves A, B, C with $B \subset A$, the second isomorphism theorem and the modular law imply that

$$\frac{A+C}{B+C} \simeq \frac{A+(B+C)}{B+C} \simeq \frac{A}{A \cap (B+C)} \simeq \frac{A}{B+(A \cap C)}.$$

Hence, we can identify the subquotients of the refined kernel filtration above as

$$\frac{(\mathcal{K}_{j+1} \cap \mathcal{I}_{\ell+1}) + \mathcal{K}_j}{(\mathcal{K}_{j+1} \cap \mathcal{I}_\ell) + \mathcal{K}_j} \simeq \frac{\mathcal{K}_{j+1} \cap \mathcal{I}_{\ell+1}}{(\mathcal{K}_{j+1} \cap \mathcal{I}_\ell) + (\mathcal{K}_j \cap \mathcal{I}_{\ell+1})}$$

Setting $i = p - \ell + j$, we get that the latter quotient is precisely $\mathcal{F}_{i,j}(M)$ (note that when $j > \ell$, the corresponding subquotient is trivial). $\square$

Lemma 2.3. *For $0 \leq j < i$, we have a natural isomorphism $\mathcal{F}_{i,j}(M) \cong \mathcal{F}_i(M)(j)$.*

Proof. For $0 < j < i$, the map $\theta_M \colon \widetilde{M} \to \widetilde{M}(1)$ induces a natural isomorphism $\mathcal{F}_{i,j}(M) \to \mathcal{F}_{i,j-1}(M)(1)$. Since $\mathcal{F}_{i,0} = \mathcal{F}_i$, the result follows by induction on j. $\square$

Remark 2.4. It follows from the proof of Proposition 2.1 that the subquotient functors $\mathcal{F}_{i,j}$ are linked as follows:

$$
\begin{tikzcd}[column sep=small, row sep=small]
\mathcal{F}_{p,p-1} \arrow[dr, dash] & & & & \\
& \mathcal{F}_{p-1,p-2} \arrow[dl, dash] \arrow[dr, dash] & & & \\
\mathcal{F}_{p,p-2} \arrow[dr, dash] & & \mathcal{F}_{p-2,p-3} \arrow[dl, dash] \arrow[dr, dash] & & \\
& \mathcal{F}_{p-1,p-3} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dl, dash] \arrow[dr, dash] & \\
\cdots \arrow[dr, dash] & \cdots & \cdots \arrow[dl, dash] \arrow[dr, dash] & \cdots & \mathcal{F}_{1,0} \arrow[dl, dash] \\
& \mathcal{F}_{p-1,1} \arrow[dl, dash] \arrow[dr, dash] & & \cdots \arrow[dl, dash] & \\
\mathcal{F}_{p,1} \arrow[dr, dash] & & \mathcal{F}_{p-2,0} \arrow[dl, dash] & & \\
& \mathcal{F}_{p-1,0} \arrow[dl, dash] & & & \\
\mathcal{F}_{p,0} & & & &
\end{tikzcd}
$$

We finish this section with an example.

Example 2.5. Let $M = kE/J^2(kE)$. Then M has constant Jordan type $[2][1]^{r-1}$. In the short exact sequence of vector bundles

$$0 \to \widetilde{M/\mathsf{Rad}M} \xrightarrow{\theta} \widetilde{\mathsf{Rad}M}(1) \to \mathcal{F}_1(M)(1) \to 0$$

the map θ (induced by θ_M) is equal to the map defining the tangent bundle (or sheaf of derivations) $\mathcal{T}$ of $\mathbb{P}^{r-1}$:

$$0 \to \mathcal{O} \to \mathcal{O}(1)^r \to \mathcal{T} \to 0.$$

It follows that $\mathcal{F}_1(M) \cong \mathcal{T}(-1)$. On the other hand we have $\mathcal{F}_{2,1}(M) \cong \mathcal{O}$, and hence $\mathcal{F}_2(M) \cong \mathcal{O}(-1)$.

3. Twists and syzygies

We need a general lemma whose proof we provide for completeness.

Lemma 3.1. *Let X be a Noetherian scheme over k, and let M, N be locally free $\mathcal{O}_X$-modules. Let $f\colon M \to N$ be a morphism of $\mathcal{O}_X$-modules such that*

$$f \otimes_{\mathcal{O}_X} k(x)\colon M \otimes_{\mathcal{O}_X} k(x) \to N \otimes_{\mathcal{O}_X} k(x)$$

is an isomorphism for any $x \in X$. Then f is an isomorphism.

Proof. It suffices to show that f induces an isomorphism on stalks. Hence, we may assume that $X = \operatorname{Spec} R$, where R is a local ring with the maximal ideal $\mathfrak{m}$, and M, N are free modules. Since specialization is right exact, f is surjective by Nakayama's lemma. Hence, we have an exact sequence of R-modules:

$$0 \to \ker f \to M \to N \to 0.$$

Since N is free, $\operatorname{Tor}_1^R(N, R/\mathfrak{m})$ vanishes, and hence $\ker f \otimes_R R/\mathfrak{m} = 0$. By Nakayama's lemma, $\ker f = 0$; therefore, f is injective. ☐

Theorem 3.2. *Let M be a finite dimensional kE-module and let $1 \leq i \leq p-1$. Then there is a natural isomorphism*

$$\mathcal{F}_i(M)(-p+i) \cong \mathcal{F}_{p-i}(\Omega M).$$

Proof. Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{\Omega M} \arrow[r] \arrow[d, "\theta_{\Omega M}"] & \widetilde{P_M} \arrow[r] \arrow[d, "\theta_{P_M}"] & \widetilde{M} \arrow[r] \arrow[d, "\theta_M"] & 0 \\ 0 \arrow[r] & \widetilde{\Omega M}(1) \arrow[r] & \widetilde{P_M}(1) \arrow[r] & \widetilde{M}(1) \arrow[r] & 0, \end{tikzcd} \tag{3.2.1}$$

where P_M is a projective cover of M. Let

$$\delta\colon \mathsf{Ker}\, \theta_M \to \mathsf{Coker}\, \theta_{\Omega M}$$

be the switchback map. A simple diagram chase in conjunction with the fact that $\theta^p_{P_M} = 0$ yields that the restriction of δ to $\mathsf{Ker}\, \theta_M \cap \mathsf{Im}\, \theta_M^{i-1}$ lands in

$$\frac{\mathsf{Ker}\, \theta^{p-i}_{\Omega M}}{\mathsf{Ker}\, \theta^{p-i}_{\Omega M} \cap \mathsf{Im}\, \theta_{\Omega(M)}}(1).$$

Projecting the latter onto

$$\mathcal{F}_{p-i,p-i-1}(\Omega M)(1) = \frac{\mathsf{Ker}\, \theta^{p-i}_{\Omega M}}{\mathsf{Ker}\, \theta^{p-i-1}_{\Omega M} + \mathsf{Ker}\, \theta^{p-i}_{\Omega M} \cap \mathsf{Im}\, \theta_{\Omega(M)}}(1),$$

we get a map of bundles:

$$\delta\colon \mathsf{Ker}\, \theta_M \cap \mathsf{Im}\, \theta^{i-1} \to \mathcal{F}_{p-i,p-i-1}(\Omega M)(1).$$

Since δ evidently kills $\mathsf{Ker}\, \theta_M \cap \mathsf{Im}\, \theta^i_M$, we conclude that δ factors through $\mathcal{F}_i(M)$. Hence, we have an induced map

$$\delta\colon \mathcal{F}_i(M) \to \mathcal{F}_{p-i,p-i-1}(\Omega M)(1).$$

A simple block count shows that this is an isomorphism at each fiber. Hence, by Lemma 3.1, this is an isomorphism of bundles. Thus using Lemma 2.3 (i.e., applying $\theta_{\Omega M}$ a further $p-i-1$ times), we have

$$\mathcal{F}_i(M) \cong \mathcal{F}_{p-i,p-i-1}(\Omega M)(1) \cong \mathcal{F}_{p-i}(\Omega M)(p-i).$$

Twisting by $\mathcal{O}(-p+i)$, we get the desired isomorphism.

Let $f\colon M \to N$ be a map of kE-modules. The naturality of the isomorphism $\mathcal{F}_i(M)(-p+i) \cong \mathcal{F}_{p-i}(\Omega M)$ is equivalent to the commutativity of the diagram

$$\begin{tikzcd} \mathcal{F}_i(M)(-p+i) \arrow[r, "\mathcal{F}_i(f)(-p+i)"] \arrow[d, "\simeq"] & \mathcal{F}_i(N)(-p+i) \arrow[d, "\simeq"] \\ \mathcal{F}_{p-i}(\Omega M) \arrow[r, "\mathcal{F}_{p-i}(\Omega f)"] & \mathcal{F}_{p-i}(\Omega N). \end{tikzcd} \tag{3.2.2}$$

The commutativity follows from the construction of the map δ and naturality of the "shifting isomorphism" of Lemma 2.3. $\square$

Corollary 3.3. *Let M be a finite dimensional kE-module and let $1 \leq i \leq p-1$. Then $\mathcal{F}_i(\Omega^2 M) \cong \mathcal{F}_i(M)(-p)$.*

Proof. Apply the theorem twice. $\square$

Corollary 3.4. *We have* $\mathcal{F}_1(\Omega^{2n}k) \cong \mathcal{O}(-np)$*, and* $\mathcal{F}_{p-1}(\Omega^{2n-1}k) \cong \mathcal{O}(1-np)$.

Proof. This follows from the theorem and the corollary, using the isomorphism $\mathcal{F}_1(k) \cong \mathcal{O}$. $\square$

Remark 3.5. If $p = 2$ then Theorem 3.2 and Corollary 3.4 reduce to the statements that $\mathcal{F}_1(\Omega M) \cong \mathcal{F}_1(M)(-1)$ and $\mathcal{F}_1(\Omega^n k) \cong \mathcal{O}(-n)$.

For a coherent sheaf $\mathcal{E}$, we denote by $\mathcal{E}^\vee = \mathcal{H}om_{\mathcal{O}}(\mathcal{E}, \mathcal{O})$ the dual sheaf.

Theorem 3.6. *Let* M^* *be the* k*-linear dual of* M*, as a* kE*-module. Then*

$$\mathcal{F}_i(M^*) \cong \mathcal{F}_i(M)^\vee(-i+1).$$

Proof. This follows from the more obvious isomorphism $\mathcal{F}_{i,i-1}(M^*) \cong \mathcal{F}_{i,0}(M)^\vee$ together with Lemma 2.3. $\square$

We finish this section with exactness properties of the functors $\mathcal{F}_i$ which will be essential in the proof of the main theorem.

Let $\mathcal{C}(kE)$ be the *exact category of modules of constant Jordan type* as introduced in [4]. This is an exact category in the sense of Quillen: the objects are finite dimensional kE-modules of constant Jordan type, and the admissible morphisms are morphisms which can be completed to a locally split short exact sequence. We call a sequence of kE-modules

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

locally split if it is split upon restriction to $k[X_\alpha]/X_\alpha^p$ for any $0 \neq \alpha \in \mathbb{A}^r$.

Proposition 3.7. *The functor* $\mathcal{F}_i \colon \mathcal{C}(kE) \to \mathrm{Coh}(\mathbb{P}_k^{r-1})$ *is exact for* $1 \le i \le p-1$.

Proof. Let

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0$$

be a locally split short exact sequence of modules of constant Jordan type. Then the Jordan type of the middle term is the sum of Jordan types of the end terms. Hence, $\mathrm{rk}\,\mathcal{F}_i(M_2) = \mathrm{rk}\,\mathcal{F}_i(M_1) + \mathrm{rk}\,\mathcal{F}_i(M_3)$ for any i. Consider the map $\mathcal{F}_i(M_2) \to \mathcal{F}_i(M_3)$. By Proposition 2.1, the specialization $\mathcal{F}_i(M_2) \otimes_{\mathcal{O}} k(\bar{\alpha}) \to \mathcal{F}_i(M_3) \otimes_{\mathcal{O}} k(\bar{\alpha})$ is surjective at any point $\bar{\alpha} \in \mathbb{P}^{r-1}$. Arguing as in Lemma 3.1, we conclude that $\mathcal{F}_i(M_2) \to \mathcal{F}_i(M_3)$ is surjective. Similarly, we show that $\mathcal{F}_i(M_1) \to \mathcal{F}_i(M_2)$ is injective. Finally, the equality $\mathrm{rk}\,\mathcal{F}_i(M_2) = \mathrm{rk}\,\mathcal{F}_i(M_1) + \mathrm{rk}\,\mathcal{F}_i(M_3)$ implies exactness in the middle term. $\square$

4. The construction

Since $H^1(E,k)$ is the vector space dual of $J(kE)/J^2(kE)$, there are elements $y_1, \dots, y_r$ forming a vector space basis for $H^1(E,k)$ and corresponding to the linear functions $Y_1, \dots, Y_r$ on $J(kE)/J^2(kE)$ introduced in Section 2. Because of the difference in structure of the cohomology ring, we divide the discussion into two cases, according as $p = 2$ or p is odd.

Case 1: $p = 2$**.** In this case the cohomology ring $H^*(E,k)$ is the polynomial algebra $k[y_1, \dots, y_r]$. We define a k-algebra homomorphism

$$\rho \colon H^*(E,k) = k[y_1, \dots, y_r] \to k[Y_1, \dots, Y_r]$$

by $\rho(y_i) = Y_i$. Recall that we have an isomorphism $\mathcal{O}(-n) = \mathcal{F}_1(k)(-n) \simeq \mathcal{F}_1(\Omega^n k)$ by Remark 3.5.

Lemma 4.1. *If $\zeta \in H^n(E,k)$ is represented by a cocycle $\hat{\zeta}\colon \Omega^{n+j}k \to \Omega^j k$ (with $j \in \mathbb{Z}$) then the diagram*

$$\begin{tikzcd} \mathcal{O}(-n-j) \arrow[r, "\rho(\zeta)"] \arrow[d, "\cong"] & \mathcal{O}(-j) \arrow[d, "\cong"] \\ \mathcal{F}_1(\Omega^{n+j}k) \arrow[r, "\mathcal{F}_1(\hat{\zeta})"] & \mathcal{F}_1(\Omega^j k) \end{tikzcd}$$

commutes.

Proof. Consider $\hat{\zeta}\colon \Omega^n k \to k$. The commutative diagram 3.2.2 applied to $\hat{\zeta}$ and iterated j times becomes

$$\begin{tikzcd} \mathcal{F}_1(\Omega^n k)(-j) \arrow[r, "\mathcal{F}_1(\hat{\zeta})(-j)"] \arrow[d, "\simeq"] & \mathcal{F}_1(k)(-j) \arrow[d, "\simeq"] \\ \mathcal{F}_1(\Omega^{n+j}k) \arrow[r, "\mathcal{F}_1(\Omega^j\hat{\zeta})"] & \mathcal{F}_1(\Omega^j k). \end{tikzcd}$$

Hence, it suffices to assume that $j = 0$. Additivity of the functor $\mathcal{F}_1$ allows us to assume that ζ is a monomial on generators $y_1, \ldots, y_r$. Finally, since multiplication in cohomology corresponds to composition of the corresponding maps on Heller shifts of k, it suffices to prove our statement for a degree one generator $\zeta = y_i$.

In the case $j = 0$, $\zeta = y_i$, we need to show that the following diagram commutes (this is the diagram above twisted by $\mathcal{O}(1)$)

(4.1.1)
$$\begin{tikzcd} \mathcal{F}_1(k) \arrow[r, "Y_i"] \arrow[d, "\simeq"] & \mathcal{F}_1(k)(1) \arrow[d, equal] \\ \mathcal{F}_1(\Omega k)(1) \arrow[r, "\mathcal{F}_1(\hat{y}_i)"] & \mathcal{F}_1(k)(1). \end{tikzcd}$$

Let E_i be the subgroup of index two in E such that y_i is inflated from E/E_i to E, namely the subgroup generated by all of $g_1, \ldots, g_r$ except g_i. Then y_i represents the class of the extension

$$0 \to k \to M_i \to k \to 0$$

where M_i is the permutation module on the cosets of E_i. This is a length two module on which $X_1, \ldots, X_r$ act as zero except for X_i, which acts as a Jordan block of length two. We have a commutative diagram of kE-modules

(4.1.2)
$$\begin{tikzcd} 0 \arrow[r] & \Omega k \arrow[r] \arrow[d, "y_i"] & P_0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & k \arrow[r] & M_i \arrow[r] & k \arrow[r] & 0 \end{tikzcd}$$

The left vertical isomorphism $\delta\colon \mathcal{F}_1(k) \xrightarrow{\sim} \mathcal{F}_1(\Omega k)(1)$ of the diagram 4.1.1 is given by the switchback map for the short exact sequence

$$0 \to \Omega k \to P_0 \to k \to 0$$

as in diagram (3.2.1). Applying θ to the commutative diagram on free $\mathcal{O}$-modules induced by the module diagram (4.1.2), we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widetilde{\Omega k} \arrow[rr] \arrow[dr] \arrow[dd, "\widetilde{y}_i"'] & & \widetilde{P_0} \arrow[rr] \arrow[dr] \arrow[dd] & & \mathcal{O} \arrow[rr] \arrow[dr] \arrow[dd, equals] & & 0 & \\
& 0 \arrow[r] & \widetilde{\Omega k}(1) \arrow[rr] \arrow[dd, "\widetilde{y}_i(1)"' near start] & & \widetilde{P_0}(1) \arrow[rr] \arrow[dd] & & \mathcal{O}(1) \arrow[rr] \arrow[dd, equals] & & 0 \\
0 \arrow[r] & \mathcal{O} \arrow[rr] \arrow[dr] & & \widetilde{M_i} \arrow[rr] \arrow[dr] & & \mathcal{O} \arrow[rr] \arrow[dr] & & 0 & \\
& 0 \arrow[r] & \mathcal{O}(1) \arrow[rr] & & \widetilde{M_i}(1) \arrow[rr] & & \mathcal{O}(1) \arrow[rr] & & 0
\end{tikzcd}$$

where all horizontal arrows going back to front are given by the operator θ on the corresponding module. The map $\widetilde{y}_i\colon \widetilde{\Omega k} \to \mathcal{O}$ is induced by $y_i\colon \Omega k \to k$. To compute the composite $\mathcal{F}_1(\hat{y}_i) \circ \delta$ we first do the switchback map of the top layer and then push the result down via $\tilde{y}_i(1)$. Since the diagram is commutative, we can first push down via the identity map of the right vertical back arrow and then do the switchback of the bottom layer. Hence, the composite $\mathcal{F}_1(\hat{y}_i) \circ \delta$ is given by the switchback map of the bottom layer; that is, of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O} \arrow[r] \arrow[d, "\theta_k"] & \widetilde{M_i} \arrow[r] \arrow[d, "\theta_{M_i}"] & \mathcal{O} \arrow[r] \arrow[d, "\theta_k"] & 0 \\
0 \arrow[r] & \mathcal{O}(1) \arrow[r] & \widetilde{M_i}(1) \arrow[r] & \mathcal{O}(1) \arrow[r] & 0
\end{tikzcd}$$

The left and right hand vertical maps here are zero. Hence, the switchback map $\delta\colon \mathcal{O} \to \mathcal{O}(1)$ is given by multiplication by θ_{M_i}, which is given by multiplication by Y_i in this situation. $\square$

Case 2: p is odd. We write $\beta\colon H^1(E,k) \to H^2(E,k)$ for the Bockstein map, and we set $x_i = \beta(y_i)$. In terms of Massey products, this is given by $x_i = -\langle y_i, y_i, \ldots, y_i\rangle$ (p terms).

The cohomology ring is a tensor product of an exterior algebra with a polynomial algebra:

$$H^*(E,k) \cong \Lambda(y_1, \ldots, y_r) \otimes_k k[x_1, \ldots, x_r].$$

We define a k-algebra homomorphism

$$\rho\colon k[x_1, \ldots, x_r] \to k[Y_1, \ldots, Y_r]$$

by $\rho(x_i) = Y_i^p$.

Lemma 4.2. *Let ζ be a degree n polynomial in $k[x_1, \ldots, x_r]$, regarded as an element of $H^{2n}(E,k)$. If ζ is represented by a cocycle $\hat{\zeta}\colon \Omega^{2(n+j)}k \to \Omega^{2j}k$ then the diagram*

$$\begin{tikzcd}
\mathcal{O}(-p(n+j)) \arrow[r, "\rho(\zeta)"] \arrow[d, "\cong"] & \mathcal{O}(-pj) \arrow[d, "\cong"] \\
\mathcal{F}_1(\Omega^{2(n+j)}k) \arrow[r, "\mathcal{F}_1(\hat{\zeta})"] & \mathcal{F}_1(\Omega^{2j}k)
\end{tikzcd}$$

commutes.

Proof. The proof is similar to the proof in the case $p = 2$, but more complicated. Again it suffices to treat the case where $\zeta = x_i$ and $j = 0$. In other words, we need to compute the composite $\mathcal{F}_1(\hat{x}_i) \circ f$ in the diagram

$$\begin{tikzcd}
\mathcal{F}_1(k) \arrow[r, "Y_i^p"] \arrow[d, "f"', "\simeq"] & \mathcal{F}_1(k)(p) \arrow[d, equal] \\
\mathcal{F}_1(\Omega^2 k)(p) \arrow[r, "\mathcal{F}_1(\hat{x}_i)"] & \mathcal{F}_1(k)(p).
\end{tikzcd}$$

where $f\colon \mathcal{F}_1(k) \to \mathcal{F}_1(\Omega^2 k)(p)$ is the isomorphism of Corollary 3.3. Let

$$0 \longrightarrow \Omega^2 k \longrightarrow P_1 \longrightarrow P_0 \longrightarrow k \longrightarrow 0$$

be a truncated projective resolution of k. Tracing through the proof of Theorem 3.2, we see that $f\colon \mathcal{F}_1(k) \to \mathcal{F}_1(\Omega^2 k)(p)$ is a composite of three maps:

(1) the switchback of the top two rows of the diagram 4.2.1 below which gives the isomorphism $\mathcal{F}(k) \to \mathcal{F}_{p-1,p-2}(\Omega k)(1)$,
(2) followed by the isomorphism $\theta^{p-2}_{\Omega k}\colon \mathcal{F}_{p-1,p-2}(\Omega k)(1) \xrightarrow{\sim} \mathcal{F}_{p-1}(\Omega k)(p-1)$ of Lemma 2.3;
(3) followed by another switchback map, now for the bottom two rows of diagram 4.2.1, which gives the isomorphism $\mathcal{F}_{p-1}(\Omega k)(p-1) \simeq \mathcal{F}_1(\Omega^2 k)(p)$.

(4.2.1)
$$\begin{tikzcd}
 & & 0 \arrow[r] & \widetilde{\Omega k} \arrow[r] \arrow[d] & \widetilde{P_0} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] & \widetilde{\Omega k}(1) \arrow[r] \arrow[d, "\theta^{p-2}_{\Omega k}"] & \widetilde{P_0}(1) \arrow[r] & \mathcal{O}(1) \arrow[r] & 0 \\
0 \arrow[r] & \widetilde{\Omega^2 k}(p-1) \arrow[r] \arrow[d] & \widetilde{P_1}(p-1) \arrow[r] \arrow[d] & \widetilde{\Omega k}(p-1) \arrow[r] \arrow[d] & 0 & & \\
0 \arrow[r] & \widetilde{\Omega^2 k}(p) \arrow[r] & \widetilde{P_1}(p) \arrow[r] & \widetilde{\Omega k}(p) \arrow[r] & 0 & &
\end{tikzcd}$$

Let E_i be the subgroup of index p such that x_i is inflated from E/E_i, namely the subgroup generated by all of $g_1, \ldots, g_r$ except for g_i. We let M_i be the permutation module on the cosets of E_i. This is a length p module on which $X_1, \ldots, X_r$ act as zero except for X_i, which acts as a Jordan block of length p. Then x_i represents the class of the 2-fold extension

$$0 \longrightarrow k \longrightarrow M_i \longrightarrow M_i \longrightarrow k \longrightarrow 0$$

where the middle map is multiplication by X_i. We construct a diagram analogous to (4.2.1) for this extension:

(4.2.2)
$$
\begin{tikzcd}
 & & 0 \arrow[r] & \widetilde{N_i} \arrow[r] \arrow[d] & \widetilde{M_i} \arrow[r] \arrow[d] & \mathcal{O} \arrow[r] \arrow[d] & 0 \\
 & & 0 \arrow[r] & \widetilde{N_i}(1) \arrow[r] \arrow[d, "\theta_{N_i}^{p-2}"] & \widetilde{M_i} \arrow[r] & \mathcal{O}(1) \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{O}(p-1) \arrow[r] \arrow[d] & \widetilde{M_i}(p-1) \arrow[r] \arrow[d] & \widetilde{N_i}(p-1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{O}(p) \arrow[r] & \widetilde{M_i}(p) \arrow[r] & \mathcal{O}(p) \arrow[r] & 0.
\end{tikzcd}
$$

Here, $N_i = \mathsf{Im}\,\{X_i \colon M_i \to M_i\}$. Just as in the proof of Lemma 4.2, the module diagram

$$
\begin{tikzcd}
0 \arrow[r] & \Omega^2 k \arrow[r] \arrow[d, "x_i"] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & k \arrow[r] & M_i \arrow[r] & M_i \arrow[r] & k \arrow[r] & 0
\end{tikzcd}
$$

induces a commutative diagram of vector bundles with (4.2.1) on top and (4.2.2) at the bottom. Arguing as in the proof of Lemma 4.1, we compute the composite $\mathcal{F}_1(\hat{x}_i) \circ f$ by first mapping the rightmost $\mathcal{O}$ of diagram 4.2.1 identically to the rightmost $\mathcal{O}$ of diagram 4.2.2, and then applying our composite of a switchback, followed by $\theta_{N_i}^{p-2}$, followed by another switchback in the diagram 4.2.2. The maps $\theta_{M_i} \colon \widetilde{M_i} \to \widetilde{M_i}(1)$ and $\theta_{N_i} \colon \widetilde{N_i} \to \widetilde{N_i}(1)$ are simply multiplication by Y_i. Since the leftmost and rightmost vertical arrows in (4.2.2) are zero, to compute the composite of the three maps involved in diagram 4.2.2, we have to multiply first by Y_i, then by Y_i^{p-2}, then by Y_i again. Hence, $\mathcal{F}_1(\hat{x}_i) \circ f = Y_i^p$. $\square$

We are ready to prove the main theorem.

Theorem 4.3. *Given any vector bundle $\mathcal{F}$ of rank s on $\mathbb{P}^{r-1}$, there exists a finitely generated kE-module M of stable constant Jordan type $[1]^s$ such that*

(i) *if $p = 2$, then $\mathcal{F}_1(M) \cong \mathcal{F}$.*
(ii) *if p is odd, then $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$, the pullback of $\mathcal{F}$ along the Frobenius morphism $F \colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$.*

Proof. Given a vector bundle $\mathcal{F}$ on $\mathbb{P}^{r-1}$, using the Hilbert syzygy theorem we can form a resolution by sums of twists of the structure sheaf:

$$0 \to \bigoplus_{j=1}^{m_r} \mathcal{O}(a_{r,j}) \to \cdots \to \bigoplus_{j=1}^{m_1} \mathcal{O}(a_{1,j}) \to \bigoplus_{j=1}^{m_0} \mathcal{O}(a_{0,j}) \to \mathcal{F} \to 0.$$

Each of the maps in this resolution is a matrix whose entries are homogeneous polynomials in $Y_1, \ldots, Y_r$. Replacing each Y_i with x_i gives matrices of cohomology elements which we may use to form a sequence of modules and homomorphisms, which takes the form

$$0 \to \bigoplus \Omega^{-\varepsilon a_{r,j}}(k) \to \cdots \to \bigoplus \Omega^{-\varepsilon a_{1,j}}(k) \to \bigoplus \Omega^{-\varepsilon a_{0,j}}(k)$$

where $\varepsilon = 1$ if $p = 2$ and $\varepsilon = 2$ if p is odd. This sequence is a complex in the stable module category $\mathsf{stmod}(kE)$. We complete the first map to a triangle whose third object we call M_{r-1}:

$$\bigoplus \Omega^{-\varepsilon a_{r,j}}(k) \to \bigoplus \Omega^{-\varepsilon a_{r-1,j}}(k) \to M_{r-1}.$$

Since the first two entries in the triangle are modules of trivial stable constant Jordan type, the same must be true for M_{r-1}. Moreover, the short exact sequence corresponding to this triangle must be locally split. Continuing by downwards induction on i from $i = r-2$ to $i = 0$ we complete triangles

$$M_{i+1} \to \bigoplus \Omega^{-\varepsilon a_{i,j}}(k) \to M_i$$

and then finally we set $M = M_0$. By construction, M_0 is a module of trivial stable constant Jordan type, and all intermediate triangles correspond to locally split sequences of modules of constant Jordan type. Applying $\mathcal{F}_1$ to this construction, we obtain an exact sequence of vector bundles by Proposition 3.7. If $p = 2$ it is isomorphic to the original resolution by Lemma 4.1 and so we have $\mathcal{F}_1(M) \cong \mathcal{F}$. On the other hand, if p is odd, each of the original matrices has been altered by replacing the variables Y_i by their pth powers by Lemma 4.2. This is the pullback of the original resolution along the Frobenius map $F\colon \mathbb{P}^{r-1} \to \mathbb{P}^{r-1}$, and so it is a resolution of $F^*(\mathcal{F})$. So in this case we have $\mathcal{F}_1(M) \cong F^*(\mathcal{F})$. □

Remark 4.4. The construction given above is not functorial, despite appearances. The problem is that given a commutative square in $\mathsf{stmod}(kG)$ in which the vertical maps are isomorphisms, it may be completed to an isomorphism of triangles, but the third arrow is not unique.

5. Chern numbers

Recall that the Chow ring of $\mathbb{P}^{r-1}$ is

$$A^*(\mathbb{P}^{r-1}) \cong \mathbb{Z}[h]/(h^r).$$

If $\mathcal{F}$ is a vector bundle on $\mathbb{P}^{r-1}$, we write

$$c(\mathcal{F}, h) = \sum_{j \geq 0} c_j(\mathcal{F}) h^j \in A^*(\mathbb{P}^{r-1})$$

for the Chern polynomial, where $c_0(\mathcal{F}) = 1$ and the $c_i(\mathcal{F}) \in \mathbb{Z}$ $(1 \leq i \leq r-1)$ are the Chern numbers of $\mathcal{F}$.

If $0 \to \mathcal{F} \to \mathcal{F}' \to \mathcal{F}'' \to 0$ is a short exact sequence of vector bundles then we have the Whitney sum formula

$$c(\mathcal{F}', h) = c(\mathcal{F}, h)c(\mathcal{F}'', h).$$

Lemma 5.1. *The formula for Chern numbers of twists of a rank s vector bundle is*

$$c_m(\mathcal{F}(i)) = \sum_{j=0}^{m} i^j \binom{s-m+j}{j} c_{m-j}(\mathcal{F}).$$

Equivalently, the total Chern class of the twists is given by

$$c(\mathcal{F}(i), h) = \sum_{n=0}^{s} c_n(\mathcal{F}) h^n (1 + ih)^{s-n} \tag{5.1.1}$$

Proof. See Fulton [9, Example 3.2.2]. □

More explicitly,

$$\begin{aligned}
c_1(\mathcal{F}(i)) &= c_1(\mathcal{F}) + is \\
c_2(\mathcal{F}(i)) &= c_2(\mathcal{F}) + i(s-1)c_1(\mathcal{F}) + i^2\binom{s}{2} \\
c_3(\mathcal{F}(i)) &= c_3(\mathcal{F}) + i(s-2)c_2(\mathcal{F}) + i^2\binom{s-1}{2}c_1(\mathcal{F}) + i^3\binom{s}{3}
\end{aligned}$$

and so on.

Lemma 5.2. *For a vector bundle $\mathcal{F}$ of rank s on $\mathbb{P}^{r-1}$ we have*

$$c(\mathcal{F},h)c(\mathcal{F}(1),h)\cdots c(\mathcal{F}(p-1),h) \equiv 1 - sh^{p-1} \pmod{(p,h^p)}.$$

Proof. We write

$$c(\mathcal{F}) = \prod_{j=1}^{s}(1+\alpha_j h),$$

where the α_j are the Chern roots. Then the formula (5.1.1) is equivalent to

$$c(\mathcal{F}(i)) = \prod_{j=1}^{s}(1+(\alpha_j+i)h).$$

Thus we have

$$c(\mathcal{F})c(\mathcal{F}(1))\cdots c(\mathcal{F}(p-1)) = \prod_{j=1}^{s}(1+\alpha_j h)(1+(\alpha_j+1)h)\cdots(1+(\alpha_j+p-1)h).$$

Now by Fermat's little theorem, we have the identity

$$x(x+y)\cdots(x+(p-1)y) \equiv x^p - xy^{p-1} \pmod{p}$$

and so putting $x = 1+\alpha_j h$, $y = h$ we obtain

$$\begin{aligned}
c(\mathcal{F})c(\mathcal{F}(1))\cdots c(\mathcal{F}(p-1)) &\equiv \prod_{j=1}^{s}((1+\alpha_j h)^p - (1+\alpha_j h))h^{p-1} \pmod{p} \\
&\equiv \prod_{j=1}^{s}(1-h^{p-1}+(\alpha_j^p-\alpha_j)h^p) \pmod{p} \\
&\equiv 1 - sh^{p-1} \pmod{(p,h^p)}.
\end{aligned}$$

A priori, this is a congruence between polynomials with algebraic integer coefficients. But if two rational integers are congruent mod p as algebraic integers then they are also congruent modulo p as rational integers. This is because their difference, divided by p, is both an algebraic integer and a rational number, therefore an integer. □

We now restate and prove Theorem 1.2.

Theorem 5.3. *Suppose that M has stable constant Jordan type $[1]^s$. Then p divides the Chern numbers $c_m(\mathcal{F}_1(M))$ for $1 \leq m \leq p-2$.*

Proof. Since M has stable Jordan type $[1]^s$, we have $\mathcal{F}_2(M) = \ldots = \mathcal{F}_{p-1}(M) = 0$. Hence, the trivial vector bundle $\widetilde{M}$ has a filtration with filtered quotients (not in order) $\mathcal{F}_1(M)$, $\mathcal{F}_p(M)$, $\mathcal{F}_p(M)(1)$, $\ldots$, $\mathcal{F}_p(M)(p-1)$. So we have

$$\begin{aligned} 1 &= c(\widetilde{M}, h) \\ &= c(\mathcal{F}_1(M), h)c(\mathcal{F}_p(M), h)c(\mathcal{F}_p(M)(1), h) \cdots c(\mathcal{F}_p(M)(p-1), h) \\ &\equiv c(\mathcal{F}_1(M), h) \pmod{(p, h^{p-1})} \end{aligned}$$

by Lemma 5.2. It follows that the coefficients $c_m(\mathcal{F}_1(M))$ are divisible by p for $1 \leq m \leq p-2$. □

Remark 5.4. For $p = 2$ this theorem says nothing. But for p odd, it at least forces $c_1(\mathcal{F}_1(M))$ to be divisible by p. As an explicit example, the twists of the Horrocks–Mumford bundle $\mathcal{F}_{\mathsf{HM}}(i)$ have $c_1 = 2i + 5$ and $c_2 = i^2 + 5i + 10$ ([11]). For $p \geq 7$ these cannot both be divisible by p, and so there is no module M of stable constant Jordan type $[1]^2$ and integer i such that $\mathcal{F}_1(M) \cong \mathcal{F}_{\mathsf{HM}}(i)$.

Remark 5.5. The conclusion of the theorem is limited to the modules of stable constant Jordan type $[1]^s$. For example, if M_n is a "zig-zag" module of dimension $2n+1$ for $\mathbb{Z}/p \times \mathbb{Z}/p$, then $\mathcal{F}_1(M_n) \simeq \mathcal{O}(-n)$ and $\mathcal{F}_1(M_n^*) \simeq \mathcal{O}(n)$ for any $n \geq 0$ (see [7, §6]).

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